Chapter 7: Gravitational Field

I Gravitational Force

Learning Objectives

- recall and use Newton's law of gravitation in the form $F = \frac{Gm_1m_2}{r^2}$

1. Newton's Law of Gravitation

Newton's law of gravitation states that two point masses attract each other with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

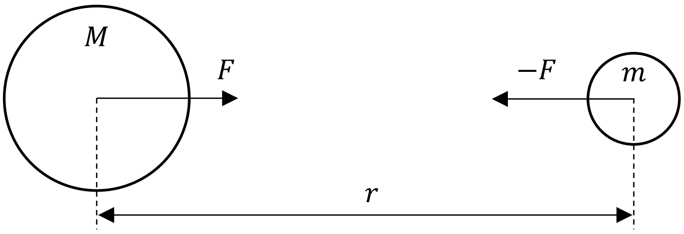
- The magnitude of the gravitational force F between two particles of masses M and m which are separated by a distance r is given by:

$$\boxed{\boldsymbol{F = \frac{GMm}{r^2}}}$$

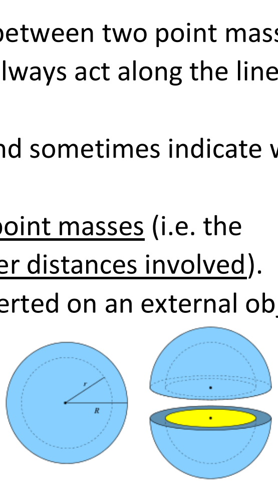
$\boldsymbol{F}$ = **gravitational force** (N)

M, m = mass (kg)

G = gravitational constant = 6.67×10^{-11} N m^2 kg^{-2} (Given)



- **Vector quantity**. Its direction towards each other.
- **SI unit**: **newton (N)**
- Note:
  - **Action and reaction pair of forces**: The gravitational forces between two point masses are equal and opposite → they are attractive in nature and always act along the line joining the two point masses.
  - **Attractive force**: Gravitational force is attractive in nature and sometimes indicate with a negative sign.
  - **Point masses**: Gravitational law is applicable only between point masses (i.e. the dimensions of the objects are very small compared with other distances involved). However, the law could also be applied for the attraction exerted on an external object by a spherical object with radial symmetry:
    - **spherical object with constant density**;
    - **spherical shell of uniform density**;
    - **sphere composed of uniform concentric shells**.



    The object will behave as if its whole mass was concentrated at its centre, and $\boldsymbol{r}$ would represent the distance between the centres of mass of the two bodies.
  - **Inverse-square law**: Newton's law of gravitation is an example of an **inverse-square law** (i.e. the force is inversely proportional to the square of the separation of the particles.

$$F \propto \frac{1}{r^2}$$

Practice 1 – Gravitational Force (L1)

Calculate the gravitation force between

(a) the Earth and moon

(b) the Earth and sun

Mass of Earth	Mass of moon	Mass of sun	Distance between	
			Earth and moon	Earth and sun
5.98×10^{24} kg	7.36×10^{22} kg	2.0×10^{30} kg	3.84×10^{8} m	1.5×10^{11} m

1.99×10^{20} N; 3.55×10^{22} N

Practice 2 – Gravitational Force (L1)

On the surface of the Earth, the gravitational force acting on an object is 45 N. When the object is at a height h above the surface, the gravitational force acting on it is 5 N. If R is the radius of the Earth, calculate the approximately value of h in terms of R.

$h = 2R$

Practice 3 – Zero Gravitational Force [ACJC/2014/Prelim/P1/15] (L1)

The Earth has approximately 81 times the mass of the Moon. There is a point between the Earth and the Moon where the resultant gravitational force on a mass m is zero. If the distance to this point from the centre of the Earth is y and from the centre of the Moon it is x, what is the ratio y/x?

$(81)^{\frac{1}{2}}$

II Gravitational Field Strength, g

Learning Objectives

- show an understanding of the concept of a gravitational field as an example of field of force and define the gravitational field strength at a point as the gravitational force exerted per unit mass placed at that point
- derive, from Newton's law of gravitation and the definition of gravitational field strength, the equation $g = \frac{GM}{r^2}$ for the gravitational field strength of a point mass
- recall and apply the equation $g = \frac{GM}{r^2}$ for the gravitational field strength of a point mass to new situations or to solve related problems
- show an understanding that near the surface of the Earth g is approximately constant and equal to the acceleration of free fall

1. Gravitational Field

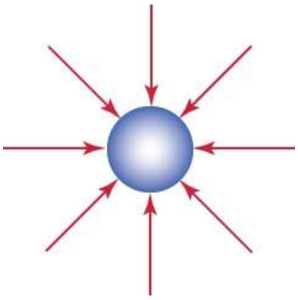
A ***gravitational field*** is a region of space in which a mass placed in that region experiences a gravitational force.

- When a mass M is placed at some point in space, it will set up a gravitational field around it. If a test mass m is placed at a distance r away from mass M, the test mass m would experience a gravitational force F exerted on it.

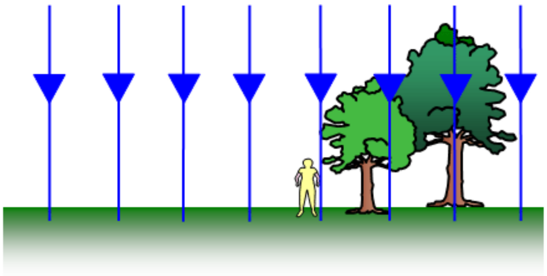
2. Gravitational Field Lines

A region of gravitational field can be visualised as an array of imaginary field lines. The gravitational force on a mass placed at a point on a field line acts along the tangent at that point.

- **Direction:**
  The gravitation field is tangent to the gravitational force acts on the point mass.
- **Magnitude**:
  The magnitude of the gravitational field is proportional to the number of lines per unit area through a surface **perpendicular** to the line.
  - A **stronger** gravitational field strength will have a closer or denser field lines.
- **Shape**:
  - For a **point mass** or a **uniform spherical mass**:
    The field lines are directed towards its centre.
  - For **near the surface of Earth**:
    If we zoom in to a region near the surface of the Earth, the field lines seem to be parallel to one another and evenly spaced → gravitational field **approximate uniform**.



point mass



Near the surface of the Earth

3. Gravitational Field Strength (g)

The ***gravitational field strength (g)*** at a point in space is defined as the gravitational force experience per unit mass at that point.

$$\boxed{g = \frac{F}{m}}$$

g = **gravitational field strength** ($\mathbf{N\ kg^{-1}}$)

F = gravitational force (N)

m = mass (kg)

- Since the gravitation force between two point masses is given by $F = GMm/r^2$:

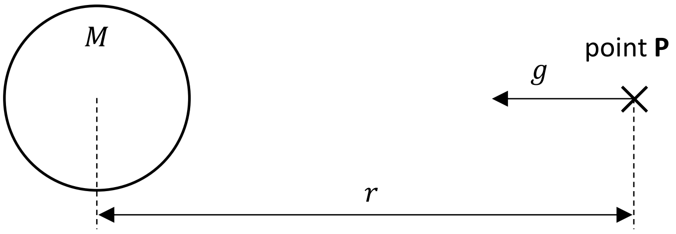
$$\boxed{g = \frac{GM}{r^2}}$$

g = **gravitational field strength** ($\mathbf{N\ kg^{-1}}$)

M = mass (kg)

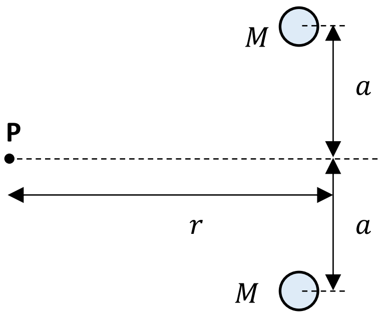
G = gravitational constant = $6.67 \times 10^{-11}\ \mathrm{N\ m^2\ kg^{-2}}$ (Given)

- **Vector quantity**. It directed towards the point mass.
- **SI unit**: **N kg^{-1}** or **m s^{-2}**.
- **Gravitational field strength** is also known as gravitational acceleration or free fall acceleration.



Practice 4 – Vector sum of Gravitational field strength (L2)

Calculate the magnitude and direction of the gravitational field at a point **P** on the perpendicular bisector of the line joining two objects of equal mass M separated by a distance $2a$ as shown in the diagram below.



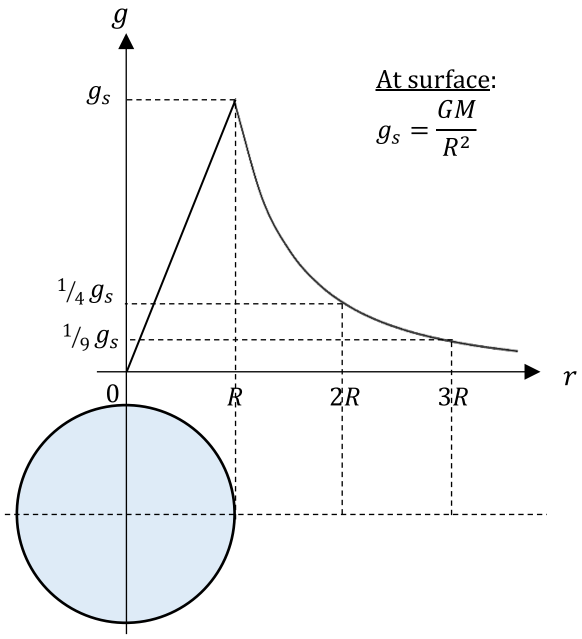
$$\frac{2GMr}{(r^2 + a^2)^{\frac{3}{2}}}$$

4. Gravitational Field Strength of a Uniform Sphere

The **shell theorem** gives gravitational simplifications that can be applied to objects inside or outside spherically symmetrical body.

- Isaac Newton proved the shell theorem and stated that:
  1. A **spherically symmetric body** affects **external objects** gravitationally as though all of its mass were concentrated at a point at its **centre**.
  2. If the body is a **spherically symmetric shell** (i.e. a hallow), no net gravitational force is exerted by the shell on any object inside, regardless of the object's location with the shell.

➢ From shell theorem, **inside** a solid sphere of constant density, the gravitational force varies linearly with distance from the centre, become zero by symmetry at the centre of mass.



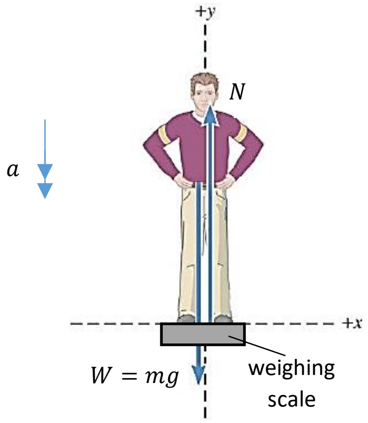
For a uniform sphere of radius R:

Inside the sphere ($r \leq R$)	**Outside the sphere ($r \geq R$)**
The mass M of the inner sphere of radius r is: $M = \rho V = \rho\left(\frac{4}{3}\pi r^3\right)$ The gravitational field strength due to the inner sphere varies linearly with distance r: $g = \frac{GM}{r^2} = \frac{G}{r^2}\left(\frac{4}{3}\rho\pi r^3\right) = \frac{4}{3}G\rho\pi r$ Hence $\boldsymbol{g} \propto \boldsymbol{r}$	The gravitational field strength due to a uniform sphere is identical to that of a point mass at the centre of the sphere: $g = \frac{GM}{r^2}$

5. Apparent weight

Apparent (or effective) weight of a person or an object is defined as the **total force** that the object **exerts on a spring scale**.

- **Apparent weight** is the normal force N exerted by the person (or object) on the scale.
- **True weight** is the gravitational force exerted on the person (or object).



Considered the force on the person, ↓:

$$mg - N = ma$$

$$\boldsymbol{N = mg - ma}$$

Apparent weight

True weight

6. Apparent weight and Earth Rotation

If the Earth is taken to be a uniform sphere, then the gravitational force F_g (or gravitational field strength g) is the same at both the equator and polar regions. However, apparent weight N will be different as the mass undergoes circular motion at the Equator.

At the polar region:

$$mg - N = F_c$$

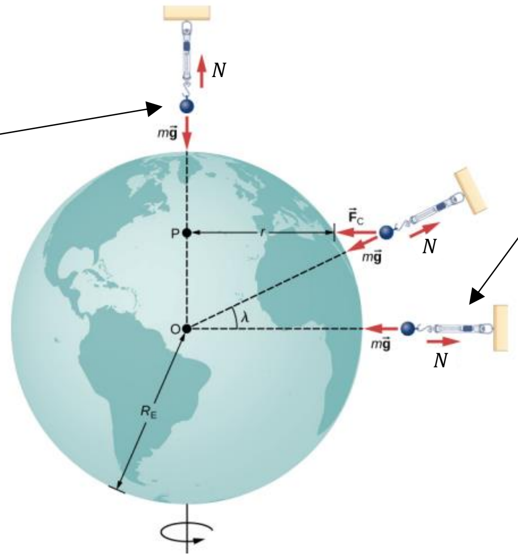
Since there is no circular motion, free fall acceleration, g_{app}:

$$F_c = 0$$

$$\therefore\ N = mg_{\text{app}} = mg$$

$$g_{\text{app}} = g$$

The free fall acceleration g_{app} is the **same** as true gravitational field strength g of the Earth at polar region.



At the equator:

$$mg - N = F_c$$

Since there is circular motion, free fall acceleration, g_{app}:

$$\therefore\ N = mg_{\text{app}} = mg - ma_c$$

$$g_{\text{app}} = g - a_c$$

where a_c is the centripetal acceleration

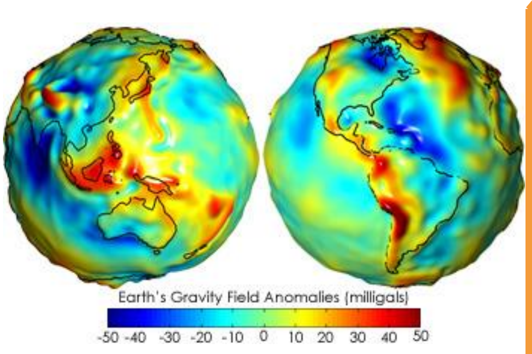
The free fall acceleration g_{app} is **less than** the true gravitational field strength g of the Earth at that region.

7. Factors affecting Gravitational Field Strength of Earth

In the above calculation, we assume the Earth is a uniform sphere, however the precise strength of Earth's gravity varies with location:

(i) Earth is not a perfect sphere: its polar diameter is about 40 km less than its equatorial diameter.

(ii) The Earth's density is not uniform. There are local variations in the Earth's gravitation field due to different terrain and composition.



Derivation from the theoretical gravity

Source: NASA GRACE mission

<u>Practice 5 – Gravitational field strength and Earth rotation</u> (L1)

The Earth may be considered to be uniform sphere of radius 6370 km, spinning on its axis with a period of 24.0 hours. The gravitational field at the Earth's surface is identical with that of a point mass of 5.98×10^{24} kg at the Earth's centre. For a 1.00 kg mass situated at the Equator,

(a) Calculate the gravitational force on the mass.
(b) Calculate the reading on an accurate spring balance supporting the mass.

9.83 N; 9.80 N

Practice 6 – Gravitational field strength and Density (L1)

The acceleration due to gravity as the earth's surface is 9.8 m s^{-2}. Calculate the acceleration due to gravity on a planet surface which has

(a) The same mass and twice the radius
(b) The same radius and twice the density
(c) Half the radius and twice the density

2.45 m s^{-2}; 19.6 m s^{-2}; 9.8 m s^{-2}

III Gravitational Potential and Potential Energy

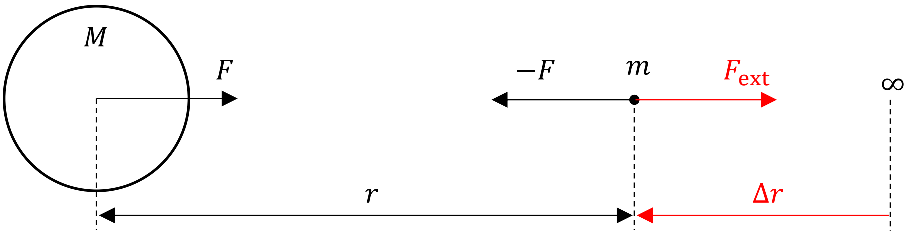
Learning Objectives

- define the gravitational potential at a point as the work done per unit mass in bringing a small test mass from infinity to that point
- solve problems using the equation $\phi = -\frac{GM}{r}$ for the gravitational potential in the field of a point mass

1. Gravitational Potential Energy, U

The ***gravitational potential energy*** **(U)** of a mass at a point in a gravitational field is defined as the work done by an external force in bringing the mass from infinity to that point.

Consider a system of two particles of masses M and m



Position of mass M is fixed and mass m is free to move. The gravitation force F between the masses is attractive → F_{ext} is negative of F_g

- At infinity, the gravitational force due to mass M is zero. The point at **infinity ($r = \infty$)** is defined to have **zero gravitational potential energy**.
- From the definition of gravitational potential energy, work done by an external force:

$$W = \int_r^\infty F_{\text{ext}}\, dr = \int_r^\infty -\frac{GMm}{r^2}\, dr = \left[\frac{GMm}{r}\right]_r^\infty = GMm\left(0 - \frac{1}{r}\right) = -\frac{GMm}{r}$$

Hence, the gravitational potential energy U of two particles of mass M and m separated by a distance r is given by

$$\boxed{U = -\frac{GMm}{r}}$$

U = **gravitational potential energy (J)**

M, m = mass (kg)

G = gravitational constant = 6.67×10^{-11} N m^2 kg^{-2} (Given)

- **Scalar quantity**.
- **SI unit: joule (J)**
- **Note:**
  - **Negative sign**: The negative sign arises because
    - **(i)** At **infinity** (reference point), the gravitational potential energy is **zero**.
    - **(ii)** Gravitational force is attractive in nature.

  ☆ As gravitational force is attractive in nature, positive work needs to be done by an external force to increase the separation between two masses. The work done by the external agent, causes the potential energy of system to increase as the distance between the masses increase. Since P.E. at infinity is set to be zero, therefore the masses when at a finite distance ($r < \infty$) have a lower potential energy than zero.
  - For **uniform gravitational field** (near the Earth): If the distance h travelled is small → $U = mgh$

- **Gravitational force F and gravitational potential energy U**:
  The gravitational force F acting on a mass m is the presence of another mass M is the **negative of the gravitational potential energy gradient**

$$\boxed{F = -\frac{dU}{dr}}$$

2. Gravitational Potential, ϕ

The ***gravitational potential*** **(ϕ)** at a point in a gravitational field is defined as the work done per unit mass by an external force in bringing a **small test mass** from infinity to that point.

$$\boxed{\phi = \frac{U}{m}}$$

$$\boxed{\phi = -\frac{GM}{r}}$$

ϕ = gravitational potential (J kg^{-1})

U = gravitational potential energy (J)

M, m = mass (kg)

G = gravitational constant = 6.67×10^{-11} N m^2 kg^{-2} (Given)

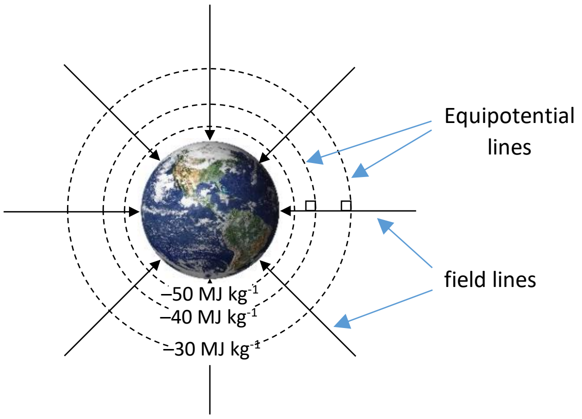
- **Scalar quantity**
- **SI unit: joule per kilogram (J kg^{-1})**
- **Note:**
  - **Negative sign**: Similar to gravitational potential energy, negative sign need to indicate (it is **not** an indication of direction).
  - **Scalar sum**: The gravitational potential at a point due to two or more masses can be found by adding the individual potentials at that point due to each mass.

3. Equipotential surface (or line)

All the points that have the same gravitational potential is called ***equipotential***.

- For point mass or uniform spherical mass:
  - All points at the same distance from the centre of the mass have the same potential.
  - The equipotential points joined together to form a circular line (2-dimension) or spherical surface (3-dimension) → **equipotential lines** or **equipotential surface**.

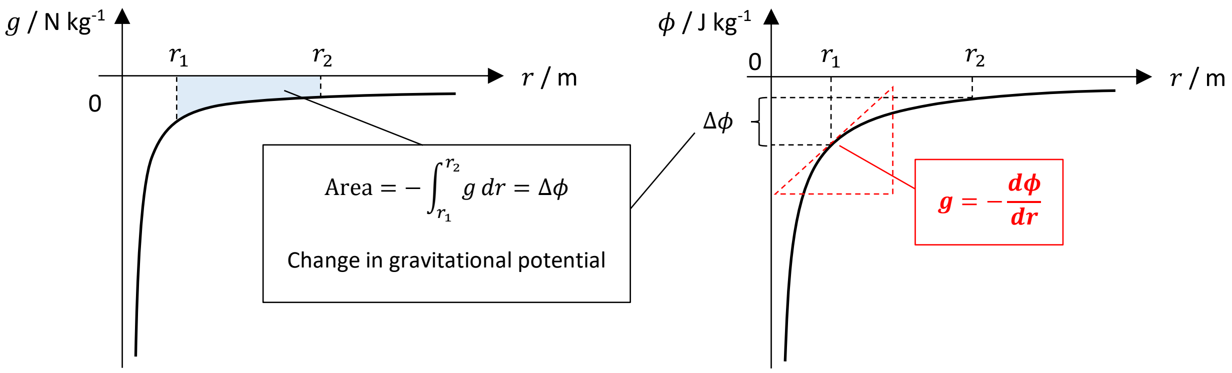
Gravitational potential at the surface of Earth:
Given: radius of Earth = 6370 km, g = 9.81 N kg^{-1}



- **Gravitational field strength g and gravitational potential ϕ**:
  The gravitational field strength g is the **negative of the gravitational potential gradient**

$$\boxed{g = -\frac{d\phi}{dr}}$$

4. Graphs of Gravitational Field Strength and Gravitational Potential



Consider 2 masses M and m (e.g. Earth and Moon), the graphs below show the variation of gravitational field strength and the corresponding gravitational potential due to M and m:

Mass of Earth	Mass of moon	Distance between Earth and moon
5.98×10^{24} kg	7.36×10^{22} kg	3.84×10^{8} m